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Turnaround radius in an accelerated universe with quasi-local mass

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Abstract. We apply the Hawking-Hayward quasi-local energy construct to obtain in a rigorous way the turnaround radius of cosmic structures in General Relativity. A splitting of this quasi-local mass into local and cosmological parts describes the interplay between local attraction and cosmological expansion.

Keywords: turnaround radius; quasi-local energy

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Contents

1 Introduction

The accelerated expansion of the universe has led cosmologists to postulate the existence of a mysterious dark energy as an explanation, a fluid with exotic properties (very negative pressure $P \simeq -\rho$, where ρ is the energy density) [1]. Alternatives to this *ad hoc* dark energy have been proposed, including the backreaction of perturbations on the cosmic dynamics [2–4], living in a giant void [5], or modifying altogether gravity at large scales [6–8]. Even if one accepts dark energy as an explanation, there are a plethora of models and it is important to design observational tests which can discriminate between dark energy and modified gravity, or between various dark energy models.

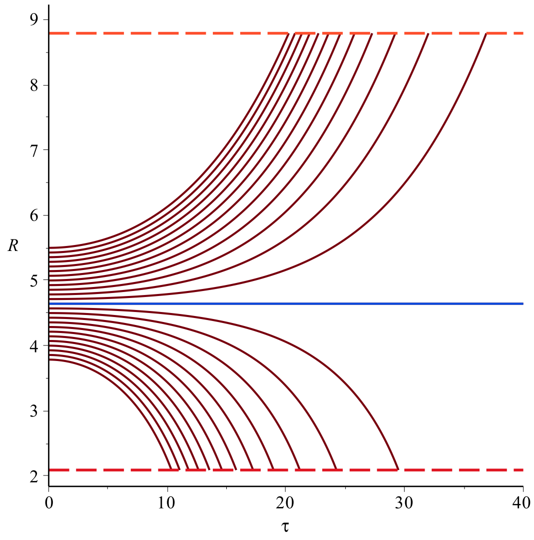
Recently, the concept of turnaround radius has been proposed as an interesting and promising way to test dark energy. Since an accelerated cosmic expansion opposes the collapse of local overdensities and the formation of cosmological structures, it is not surprising that, in an accelerating Friedmann-Lemaître-Robertson-Walker (FLRW) universe with a *spherical* inhomogeneity, there is an upper bound to the radius of an overdensity of a certain mass, beyond which this matter cannot collapse but can only expand. Assuming, for reference, that the universe in which we live is approximately described by the standard Λ-Cold Dark Matter (ΛCDM) scenario of General Relativity with a cosmological constant or dark energy, it has been suggested that the ΛCDM scenario can be tested by studying whether the mass-radius relation of structures in the cosmos respects a theoretical prescription [22–32].

To be more precise, consider a spherical configuration of mass in an accelerating FLRW universe: when the outer layers of this material configuration reach zero radial acceleration and collapse under the self-gravity of the perturbation, it is said that the perturbation has reached its *turnaround radius* ([31, 32], see also [33–39]). In a decelerated universe there is no upper bound on the turnaround radius, but in a de Sitter background with constant Hubble parameter $H = \sqrt{\Lambda/3}$ (where $\Lambda > 0$ is the cosmological constant), there exists the upper bound

$$R_c = \left(\frac{3GM}{\Lambda}\right)^{1/3} \tag{1.1}$$

on the turnaround radius, where R is the areal radius of the Schwarschild-de Sitter (SdS) line element,

$$ds^2 = -\left(1 - \frac{2M}{R} - H^2 R^2\right) dt^2 + \frac{dR^2}{1 - \frac{2M}{R} - H^2 R^2} + R^2 d\Omega_{(2)}^2 \tag{1.2}$$



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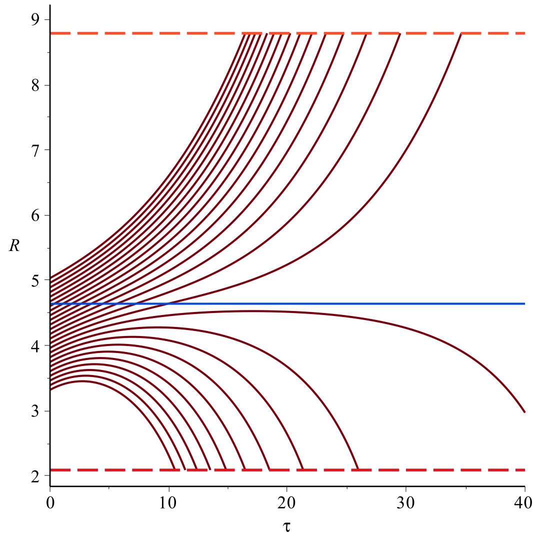

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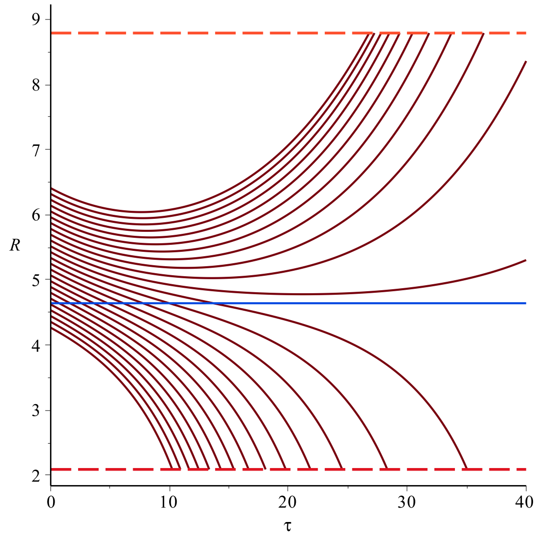


(c)

Figure 1. Radial trajectories of massive test particles in the SdS spacetime subject to zero 1(a), outgoing 1(b) and in-falling 1(c) initial velocities. The solid blue line is the location of the critical radius R_c (1.1). To make these figures we used the illustrative parameter values $M = 1$ and $H = 0.1$ for which both the cosmological and black hole horizons are present (dashed lines). τ is the proper time (an affine parameter) along the geodesics.

in static coordinates, $d\Omega^2_{(2)} = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the line element on the unit 2-sphere, and $H = \sqrt{\Lambda/3}$ [31, 32]. In Fig.1 we plot the trajectories of test particles placed at various radii in the SdS spacetime, with initially zero 1(a), outgoing 1(b) and in-falling 1(c) initial velocities. The figure shows that the critical radius is not an absolute boundary in the same way a horizon is since timelike geodesics can cross it in either direction. The critical radius is that point beyond which, if you cross outside of it in geodesic motion, you will never cross back without proper acceleration, and *vice-versa*. This one-way property is also directly visible

from the geodesic equation for radial timelike trajectories, which can be shown to be given by

$$\ddot{R}(\tau) = \left(R^3 - R_c^3\right) \frac{H^2}{R^2}, \tag{1.3}$$

where an overdot denotes differentiation with respect to the proper time τ parametrizing timelike geodesics. We see that for $R > R_c$ we have $\ddot{R} > 0$ and for $R < R_c$ we have $\ddot{R} < 0$. This is the defining property of the turnaround radius.[1]

The concept of turnaround radius can be generalized to arbitrary (but accelerating) FLRW universes [32]. In Ref. [32], also a Lemaître-Tolman-Bondi metric

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+f(r)} dr^2 + R^2(t,r) d\Omega_{(2)}^2 \tag{1.5}$$

is studied, where $f(r)$ is an arbitrary function related to an initial density profile, and a cosmological constant Λ is present in addition to the usual dust fluid (a prime denotes differentiation with respect to r). The authors study shells of areal radius R and derive their radial acceleration

$$\ddot{R} = -\frac{G\mathcal{M}(r)}{R^2} + \frac{\Lambda R}{3}, \tag{1.6}$$

obtaining a turnaround radius

$$R_c(t_c, r_c) = \left(\frac{3G\mathcal{M}(r_c)}{\Lambda}\right)^{1/3}. \tag{1.7}$$

Here $\mathcal{M}(r) = 4\pi \int_0^R R^2 \rho dR$ is the well-known Lemaître mass and ρ is the density of the cosmic dust in the Lemaître-Tolman-Bondi spacetime.

The Lemaître-Tolman-Bondi model is only one of the possible choices to describe a spherical inhomogeneity embedded in a FLRW universe: other choices exist, for example the McVittie [10] and generalized McVittie [9] metrics, which have been the subject of recent attention [11–21] and which describe a generalization of Schwarzschild-de Sitter spacetime to include a time-dependent Hubble parameter $H(t)$. The McVittie spacetime is described by the metric

$$ds^2 = -\left(1 - \frac{2m}{R} - H^2R^2\right) dt^2 + \left(1 - \frac{2m}{R}\right)^{-1} dR^2 - \frac{2HR}{\sqrt{1-\frac{2m}{R}}} dt dR + R^2 d\Omega_{(2)}^2 \tag{1.8}$$

with $H(t) \equiv \dot{a}(t)/a(t)$ the time-dependent Hubble parameter. The radial geodesic equation in McVittie spacetime is a generalization of eq. (1.3) and it can be shown to be

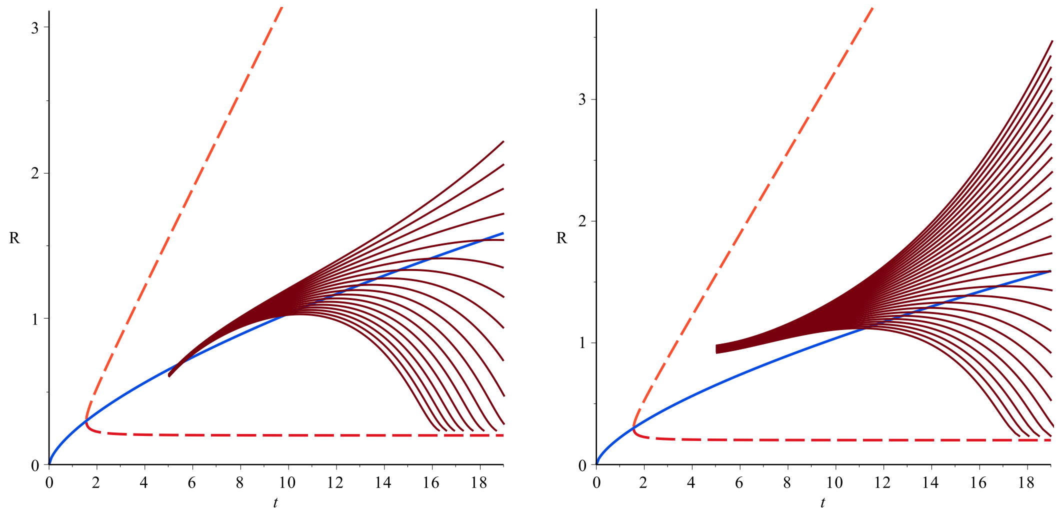
$$\ddot{R}(\tau) = \left(R^3 - R_c^3\right) \frac{H^2}{R^2} + \dot{H} R \sqrt{1 - \frac{2m}{R}}\, \dot{t}^2(\tau) \tag{1.9}$$

Note that in the de Sitter case (when $a(t) = \exp(\mathcal{H}t)$ with $\mathcal{H}$ constant), the extra term proportional to $\dot{t}^2$ vanishes exactly. Therefore, the turnaround property at $R = R_c$ is not

[1]The other geodesic equation for radial motion is

$$\ddot{t}(\tau) - 2\left(R^3 - R_c^3\right) \frac{H^2}{R^2} \left(1 - \frac{2M_{\rm MSH}}{R}\right)^{-1} \dot{t}(\tau)\dot{R}(\tau) = 0, \tag{1.4}$$

where $M_{\rm MSH}$ is the quasi-local mass of a sphere of radius R.



(a) (b)

Figure 2. Timelike geodesics with large positive (subfigure 2(a)) and small positive (subfigure 2(b)) initial velocities. The critical radius is the solid blue line and the apparent horizons for McVittie spacetime with $a(t) = t^3$ are the dashed red and orange curves. t is the coordinate time.

an exact result in the McVittie case and the details depend on the precise evolution of the background as well as the time in which the sprinkling takes place. For power-law expansion $a(t) = t^p$ at the radius $R = R_c$ we have

$$\ddot{R}(\tau, R_c) \propto \frac{-1}{t^{8/3}(\tau)} \sqrt{1 - \frac{2m}{R_c}}\, \dot{t}^2(\tau), \tag{1.10}$$

which goes to zero at late times, establishing that the sphere $R = R_c$ is critical at late times. In Fig. 2 we show congruences of timelike geodesic trajectories starting near the time-dependent radius $R = R_c$ and the (time-dependent) apparent horizons to illustrate this discussion for McVittie. The figure shows that it is possible to find geodesics which escape but are subsequently captured by the critical sphere. Note that the result (1.10) shows that the exact critical radius is strictly larger than R_c since $\ddot{R}(t, R_c)$ is always negative inside the horizons.

In any case, when one decides to describe a large-scale structure (assumed to be spherical for simplicity) in a FLRW universe, the deviations of the metric from an exact FLRW one are small (although the density contrast, *i.e.*, the ratio between the density of the local perturbation and the average cosmological density can be large). By adopting the conformal Newtonian gauge, the perturbed FLRW metric is written as

$$ds^2 = a^2(\eta) \left[-(1 + 2\phi) d\eta^2 + (1 - 2\phi) \left(dr^2 + r^2 d\Omega^2_{(2)} \right) \right] , \tag{1.11}$$

where η is the conformal time of the FLRW background (defined by $dt = a d\eta$ in terms of the comoving time t), a is the scale factor, and ϕ is a post-Friedmannian potential describing the local inhomogeneity. Here we neglect vector and tensor perturbations, as is customary to first order, and we restrict our discussion to linear order in the metric perturbations, which are assumed to be small everywhere. As usual, the spatial derivatives of ϕ dominate over its time

derivative, and we confine ourselves to General Relativity (otherwise two distinct potentials ψ and ϕ would appear in the metric coefficients g_{00} and g_{11}).

In spherical symmetry $\phi = \phi(r)$ and it is possible to introduce the areal radius

$$R(\eta, r) = a(\eta) r \sqrt{1 - 2\phi(r)} \,. \tag{1.12}$$

The authors of [32] derive the radial acceleration of a spherical shell in comoving time

$$\ddot{R} = -\frac{4\pi}{3}(\rho_E + 3P_E)R - \frac{G\mathcal{M}(r)}{R^2} = \frac{\ddot{a}}{a}R - \frac{G\mathcal{M}(r)}{R^2} \,, \tag{1.13}$$

where an overdot denotes differentiation with respect to the comoving time of the FLRW background, ρ_E and P_E are the energy density and pressure of the dark energy propelling the cosmic acceleration, respectively,

$$\mathcal{M}(r) = 4\pi \int_0^r r'^2 \rho(r') dr' \tag{1.14}$$

is the mass contained within the shell of radius $R(t, r)$, and the energy density ρ includes both the homogeneous cosmological density ρ_E and the inhomogeneous density of the perturbation.[2] It is then straightforward to derive the radius R_c corresponding to the situation in which the two contributions, cosmological and local, cancel out in the right hand side of eq. (1.13) [32],

$$R_c = \left(-\frac{3\mathcal{M}}{4(1+3w)\pi\rho_E} \right)^{1/3} \,, \tag{1.15}$$

where $w \equiv \frac{P_E}{\rho_E} < -\frac{1}{3}$ is the equation of state parameter of dark energy. This expression is taken as the turnaround radius in a dark energy-dominated FLRW universe in which $H = H(t)$. For $w = -1$ (the value corresponding to a de Sitter background), it reduces to the expression (1.1). A similar expression defines the "sphere of influence" used for the same purpose in the literature (*e.g.*, [41]).

Writing the perturbed FLRW metric in terms of the areal radius and the Misner-Sharp Hernandez quasi-local mass[3]

$$M_{\mathrm{MSH}} = -\phi R + \frac{H^2 R^3}{2} - H^2 R^3 \phi \tag{1.16}$$

we have

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2M_{\mathrm{MSH}}}{R}\right) dt^2 + \left(1 - \frac{2M_{\mathrm{MSH}}}{R} + H^2R^2\right)^{-1} dR^2 + R^2 d\Omega^2_{(2)} \\ &- \frac{2HR}{1 - \frac{2M_{\mathrm{MSH}}}{R} + H^2R^2} dR dt \end{aligned} \tag{1.17}$$

which is "McVittie-like", since the McVittie metric is written in terms of the quasi-local mass as

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2M_{\mathrm{MSH}}}{R}\right) dt^2 + \left(1 - \frac{2M_{\mathrm{MSH}}}{R} + H^2R^2\right)^{-1} dR^2 + R^2 d\Omega^2_{(2)} \\ &- \frac{2HR}{\sqrt{1 - \frac{2M_{\mathrm{MSH}}}{R} + H^2R^2}} dR dt \,. \end{aligned} \tag{1.18}$$

[2] However, this expression for the mass is not written explicitly in [32].

[3] We will introduce and motivate the quasi-local mass more fully in section 2. Essentially, M_{MSH} measures the total mass/energy inside a sphere of areal radius R, including gravitational contributions.

We see that the only difference is in the off-diagonal terms in the metric which could arguably be equal to the level of approximation we employ in the FLRW metric.[4] As a linearized solution, however, the perturbed FLRW metric in "McVittie form" should only be used far from the inner apparent horizon when it exists. Nevertheless, the geodesic structure near the critical radius and its approximate nature carry over from the McVittie case.

We note that:

1. There is a vast literature on the interplay between cosmological expansion and local physics (see [42] for a review), originating in the "pure" relativity community, which seems to be mostly neglected by the more astrophysically-oriented community working on the turnaround radius and studying real celestial objects. Here we attempt to bridge between these two approaches and communities, in view of the fact that the turnaround radius holds much promise for testing the ΛCDM model.

2. Discussions relying on a particular gauge (such as the conformal Newtonian gauge) are gauge-dependent and one would like, instead, to have gauge-invariant characterizations.

3. The "mass contained within a sphere" is defined in a rather hand-waving way in the literature on the turnaround radius: should this "mass" include that of the cosmological fluid but not its pressure (as in [32])? Should it include the negative pressure of dark energy? If not, why not? Should it include only the local density (*i.e.*, energy density minus the cosmological density)? These questions are not answered, nor asked explicitly, in the literature on the turnaround test of ΛCDM. It is clear, however, that the turnaround spheres considered contain significant amounts of dark energy and a relativistic definition of gravitating mass-energy in them should be used. In other words, the universe of cosmology is relativistic and regions sufficiently large should *a priori* be treated with relativity and not with Newtonian physics. Indeed, the questions above are not trivial: researchers in General Relativity have wondered for decades about the correct definition of gravitating mass-energy for a given spacetime, especially when the latter is not asymptotically flat (as is the case in cosmology), and have come up with several quasi-local approaches and mass definitions (see [43] for a review). There is now a large consensus that the "correct" notion is the Hawking-Hayward quasi-local mass [44, 45]. We have recently applied [47] the Hawking-Hayward quasi-local mass M_{HH} construct to cosmological perturbations, giving a covariant splitting of this quantity into "local" and "cosmological" parts which can be used to separate the gravitational effects of local perturbations and cosmological background. Although Ref. [47] studied the growth of structures in the dust-dominated era, the formalism presented in it is general and can be immediately adapted to the accelerated era, which is what we set out to do here. Ref. [47] is not restricted to spherical symmetry, although it includes the spherically symmetric case as an example.

Our goals include clarifying the issue of mass within a turnaround sphere and giving firmer foundations to the concept of turnaround radius. Although, in practice, there is at least a $10-30$ % uncertainty in the determination of masses and radii of the astronomical objects used in the turnaround test of ΛCDM, vagueness and uncertainty in the theory do not help getting firm outcomes for the test.

[4] Since $M_{\mathrm{MSH}}=\mathcal{O}(\phi)$ we have $H^2R^2=\mathcal{O}(\phi)$ and hence the numerator is $\mathcal{O}\left(\phi^{1/2}\right)$ and the denominator can be split into two products of the square root of itself, one of which contributes $1+\mathcal{O}(\phi)$ to the numerator and is therefore discarded.

It turns out that, in a FLRW universe dominated by dark energy with equation of state parameter $w \simeq -1$, the difference between the more rigorously defined turnaround radius and the one used in the astronomical literature is minimal. The ambiguities inherent in the definition of "mass" are clarified (although the astronomical uncertainties on the *values* of these masses, of course, persist). The "mass" is here defined in a completely gauge- and coordinate-independent way (independent of the conformal Newtonian gauge chosen at the outset for the perturbed FLRW metric).

2 A rigorous definition of turnaround radius

Here we approach the issue of cosmological accelerated expansion versus local physics and that of a rigorous definition of turnaround radius using the Hawking-Hayward quasi-local mass. Our assumptions are:

1. General Relativity is valid (otherwise the Hawking-Hayward quasi-local energy is not defined and the perturbed FLRW metric is not given by eq. (2.1)).

2. We restrict ourselves to first order in the metric perturbations (however, density fluctuations can be large). The perturbed FLRW metric is written in the Newtonian conformal gauge as
$$ds^2 = a^2(\eta)\left[-(1+2\phi)d\eta^2 + (1-2\phi)\left(dr^2 + r^2 d\Omega^2_{(2)}\right)\right], \tag{2.1}$$
but our final results are gauge-independent. We neglect vector and tensor perturbations in the metric to this order, and spatial derivatives $\partial_i\phi$ of the post-Friedmannian perturbation potential are assumed to dominate over the time derivative $\partial_t\phi$.

3. The background is assumed to be a spatially flat FLRW universe dominated by a single dark energy fluid with energy density ρ_E, pressure P_E and equation of state parameter $w \equiv \frac{P_E}{\rho_E} < -\frac{1}{3}$ (until stated explicitly, we do not assume that w is constant in time or redshift).

4. The spacetime is spherically symmetric, $\phi = \phi(r)$. This last assumptions recurs in previous literature [31, 32] and is regarded here only as a simplification to be relaxed at a later stage, whose effects are discussed in [51].

These assumptions exclude the alternative explanations of the present acceleration of the universe mentioned in the Introduction and the present discussion is to be seen as a test of the ΛCDM model only. For example, even remaining within the context of General Relativity, if backreaction of inhomogeneities is invoked to explain the cosmic acceleration, the background metric is affected and is no longer given by eq. (2.1), and an effective turnaround radius would have to be introduced through an averaging of the physical variables (which we leave for future work).

2.1 Hawking-Hayward quasi-local mass and its local and cosmological parts

The physical, gravitating mass of a non-asymptotically flat spacetime has been the subject of much debate in General Relativity and, after extensive work on quasi-local energies [43], the community seems to have settled on the Hawking-Hayward construct [44, 45]. There is little doubt that when a region of spacetime of size not entirely negligible in comparison with the Hubble radius H^{-1} is considered, a relativistic (as opposed to Newtonian) mass-energy

causing gravitational effects needs to be employed. The Hawking-Hayward quasi-local mass in defined as follows [44–46]. Let S be a closed, orientable, 2-dimensional surface. Let $\mathcal{R}$ be the induced Ricci scalar on S and consider the outgoing and ingoing congruences of null geodesics from S, which have expansion scalars $\theta_\pm$ and shear tensors $\sigma_{ab}^{(\pm)}$, respectively. Let ω^a be the projection onto S of the commutator of the null normal vectors to S ("anholonomicity"), then the quasi-local mass of S is

$$M_{\rm HH} \equiv \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}}\int \mu \left(\mathcal{R} + \theta_+\theta_- - \frac{1}{2}\sigma_{ab}^{(+)}\sigma^{ab}_{(-)} - 2\omega_a\omega^a \right) , \tag{2.2}$$

where μ is the volume 2-form of S and A is its area [44, 45].

In the geometry (2.1), the Hawking-Hayward mass of a surface S corresponding to a sphere in the unperturbed FLRW background was computed, to first order in the perturbations, in Ref. [47]. The result becomes particularly simple in spherical symmetry in which the Hawking-Hayward mass reduces [46] to the better known Misner-Sharp-Hernandez mass [49, 50]. The result is [47]

$$M_{\rm HH} = ma + \frac{H^2R^3}{2}(1-\phi) \tag{2.3}$$

where m is the Newtonian mass obtained by integrating ϕ over a sphere, $m = \int \nabla^2\phi \, d^3\vec{x}$, and is constant. The physical mass scale of the perturbation described by ϕ, like physical length scales, is obtained by multiplying by the scale factor (more on this later). In units in which $G = c = 1$, $2m$ gives the comoving Schwarzschild radius scale of the perturbation, while the physical scale is $2ma$. Since $\phi \simeq -m/r \simeq -ma/R$ to first order, where

$$R = ar\sqrt{1-2\phi} \approx ar(1-\phi) \tag{2.4}$$

is the areal radius of the perturbed FLRW space in spherical symmetry, we can make the approximation

$$M_{\rm HH} \simeq ma + \frac{H^2R^3}{2} \tag{2.5}$$

to this order. Note that the factor HR, the ratio of the size of a sphere of areal radius R to the Hubble radius H^{-1}, is small for the structures that we will consider, and it appears to the second power in eq. (2.3). The decomposition (2.5) splits the Hawking-Hayward/Misner-Sharp-Hernandez mass into a "local" part ma and a "cosmological" part $H^2R^3/2$. It was shown in [47] that this decomposition is covariant and gauge-invariant to first order, in spite of starting with the particular gauge (2.1).

Eq. (2.5) is particularly suited for quantifying the competition between cosmological expansion and local physics, since the gravitational effects are due to the physical gravitating masses of the local structure and of the surrounding cosmology. The criterion defining the *critical* (turnaround) radius for a system on the verge of breaking down is now that for such a system the two contributions to $M_{\rm HH}$ are equal,

$$ma = \frac{H^2R^3}{2} \tag{2.6}$$

which yields

$$R_c(t) = \left(\frac{2ma}{H^2}\right)^{1/3} \tag{2.7}$$

for the critical radius. With this definition there are no ambiguities in the concept of mass contained in the sphere of radius R_c (more about this later). Note that eq. (2.6) equating the two contributions to $M_{\rm HH}$ makes it explicit that we are in an intermediate regime in which local gravitational effects are comparable with cosmological ones. This is the "gray" area lying in between the Newtonian regime $R \ll H^{-1}$ in which cosmological effects are negligible, and the FLRW regime $R \sim H^{-1}$ in which relativistic cosmology dominates. By using the Hamiltonian constraint

$$H^2 = \frac{8\pi}{3}\rho_E \tag{2.8}$$

for a spatially flat FLRW background dominated by dark energy, one obtains

$$R_c(t) = \left(\frac{3ma}{4\pi\rho_E}\right)^{1/3} . \tag{2.9}$$

Let us assume now that the equation of state parameter w is constant; then using the relations valid for the FLRW background

$$a(t) = a_* t^{\frac{2}{3(1+w)}} , \qquad \rho_E(t) = \frac{\rho_0}{a^{3(1+w)}} \qquad (w \neq -1) , \tag{2.10}$$

where a_* and ρ_0 are constants, the critical radius is seen to vary with the scale factor as

$$R_c(a) = \left(\frac{3m}{4\pi\rho_0} a^{3w+4}\right)^{1/3} \equiv R_0\, a^{\frac{3w+4}{3}} \tag{2.11}$$

(which makes it clear that the "critical sphere" is not comoving) and where the constant R_0 is

$$R_0 \equiv \left(\frac{3m}{4\pi\rho_0}\right)^{1/3} = \left[\frac{9m(1+w)^2}{2a_*^{3(1+w)}}\right]^{1/3} , \tag{2.12}$$

where the relation

$$H = \frac{2}{3(1+w)t} = \frac{2a^{\frac{3(1+w)}{2}}}{3(1+w)a_*^{\frac{3(1+w)}{2}}} \tag{2.13}$$

has been used. By showing explicitly the physical local mass ma, one has instead

$$R_c(a) = \left(\frac{3ma}{4\pi\rho_0}\right)^{1/3} a^{w+1} . \tag{2.14}$$

As a function of time, the critical radius is

$$R_c(t) = \left(R_0 a_*^{\frac{3w+4}{3}}\right) t^{\frac{2(3w+4)}{9(w+1)}} \tag{2.15}$$

for $w \neq -1$. For $w = -1$, corresponding to a de Sitter background, it is instead

$$R_c(t) = R_0 a^{1/3} = R_0\, a_*^{1/3}\, \mathrm{e}^{H_0 t/3} . \tag{2.16}$$

The exponent $\frac{2(3w+4)}{9(w+1)}$ in eq. (2.15) is

- positive if $w < -\frac{4}{3}$ or $w > -1$,

- negative if $-\frac{4}{3} < w < -1$,

and vanishes in a phantom universe with $w = -4/3$, in which R_c remains constant (a rather odd occurrence). We can express R_c as a function of redshift z using $a_0/a = z + 1$, where a_0 is the present value of the scale factor. The result is

$$R_c(z) = \left[\frac{3a_0^{3(w+1)}}{4\pi\rho_0} ma\right]^{1/3} \frac{1}{(z+1)^{w+1}} = \frac{R_c(0)}{(z+1)^{w+4/3}} \tag{2.17}$$

if we insist on using ma as the local mass, where

$$R_c(0) = \left(\frac{3m\, a_0^{3w+4}}{4\pi\rho_0}\right)^{1/3} . \tag{2.18}$$

By setting the present value a_0 of the scale factor to unity, as customary, one obtains the equation of state parameter of dark energy as a function of the redshift and the mass ma

$$w(z) = -1 + \frac{1}{\ln(z+1)} \left\{ \ln\left[\left(\frac{3ma}{4\pi\rho_0}\right)^{1/3} \frac{1}{R_c(z)}\right]\right\} , \tag{2.19}$$

which can be used to constrain w if ma and R_c are known.

The Hawking-Hayward mass of a "critical sphere" of radius R_c is

$$M_c \equiv M_{\mathrm{HH}}(R_c) = ma + \frac{H^2 R_c^3}{2} = 2ma = H^2 R_c^3 = \frac{8\pi}{3}\rho R_c^3 \tag{2.20}$$

by definition, and it includes both the "local" mass ma due to the perturbation and the contribution of the energy density of the cosmic fluid. When m is constant, $M_{\mathrm{HH}}(R_c)$ is "comoving", while R_c is not.

2.2 Comparison with the previous definition of R_c

We are now ready to compare our $R_c(t)$ with previous definitions. Since the "mass" $\mathcal{M}(r)$ contained in a sphere of areal radius R is not clearly defined in previous literature, care must be taken in this comparison. The authors of [32] obtain

$$R_c^{(1)} = \left(\frac{3\mathcal{M}(r)}{4\pi|1+3w|\rho_E}\right) \tag{2.21}$$

where it is argued that $\mathcal{M}(r) = 4\pi\int_0^r r'^2\rho(r')dr'$ and ρ includes the homogeneous energy density (but not the pressure) of dark energy and the density of the perturbation. Based on our previous discussion, this would imply the correspondence $\mathcal{M} \leftrightarrow 2ma$ and

$$\frac{R_c}{R_c^{(1)}} = \left(\frac{|1+3w|}{2}\right)^{1/3} . \tag{2.22}$$

For $w \approx -1$ this ratio is almost unity. Therefore, our calculation can be regarded as a rigorous justification of the expression of R_c previously obtained in [32] (with a small correction), plus an important clarification of the meaning of "mass in the sphere of radius R_c". Since the idea is to constrain w, however, it is best not to assume the value of this quantity and to use eq. (2.9) for the value of R_c and eq. (2.22) for its relation with previous literature.

If the equation of state parameter w is not constant, using the covariant conservation equation

$$\dot{\rho}_E + 3H(P_E + \rho_E) = 0\,, \tag{2.23}$$

differentiating the relation $a_0/a = z+1$ to obtain $dt = -\frac{a_0 dz}{\dot{a}(z+1)^2}$, and integrating eq. (2.23) yield

$$\int dz\, \frac{w(z)+1}{z+1} = \ln\left[\left(\frac{3ma}{4\pi\rho}\right)^{1/3} R_c^{-1}\right] \tag{2.24}$$

which, for $w =$ constant, reduces to eq. (2.19).

3 Dynamics of the critical sphere

Let us consider the "critical sphere" of radius $R_c(t)$, which contains a system on the verge of breaking down under the influence of the cosmic acceleration. Without assuming $w =$ constant, the differentiation of eq. (2.9) yields, in conjunction with the covariant conservation equation (2.23),

$$\frac{\dot{R}_c}{R_c} = \left(w + \frac{4}{3}\right) H\,, \tag{3.1}$$

which quantifies the deviation of the critical sphere from the comoving motion of the cosmic substratum. In an accelerated universe with $w < -1/3$, the critical sphere grows slower than the cosmic substratum, while it grows faster in a decelerated universe in which it is easier for large structures to collapse. By further assuming $w =$ const. and using the scale factor (2.10), eq. (3.1) becomes

$$\frac{\dot{R}_c}{R_c} = \frac{2(w+4/3)}{3(w+1)}\,\frac{1}{t} \tag{3.2}$$

for $w \neq -1$, which tends to zero as $t \to +\infty$ and integrates to

$$R_c(t) = R_* t^{\frac{2(w+4/3)}{3(w+1)}}\,. \tag{3.3}$$

In the de Sitter case $w = -1$, instead,

$$R_c(t) = R_* \exp\left[\left(w + \frac{4}{3}\right) H_0 t\right] \tag{3.4}$$

where $H_0 \equiv \sqrt{\Lambda/3} =$ const.

The physical volume V_c of the critical sphere, of course, obeys

$$\frac{\dot{V}_c}{V_c} = 3\,\frac{\dot{R}_c}{R_c} = (3w+4)\,H\,. \tag{3.5}$$

The Hawking-Hayward/Misner-Sharp-Hernandez quasi-local mass contained in the critical sphere is

$$M_c = \frac{8\pi}{3}\rho_E R_c^3 = H^2 R_c^3 \tag{3.6}$$

and its time derivative is easily found to be

$$\dot{M}_c = (HR_c)^3 = \frac{6ma}{H^{-1}} \tag{3.7}$$

using eqs. (2.23) and (3.1) and without assuming $w = \text{const}$. Thus, R_c is the fraction $(\dot{M}_c)^{1/3}$ of the Hubble radius H^{-1}. Taking the ratio of eqs. (3.7) and (3.6) yields again the result that M_c is comoving, $\dot{M}_c/M_c = H$, which we already know since it was previously established that $M_c = 2ma$.

4 Conclusions

The mathematical splitting of $M_{\rm HH}$ gives nearly the same numerical result as previous literature for the turnaround radius. This is reassuring since it inspires confidence in the use of the formal quantity $M_{\rm HH}$ in astrophysics, because its application reproduces almost exactly results obtained with less formal reasoning based on the motion of test and gravitating particles and fluids, but puts those results on a firmer theoretical ground. Thus far (with the exception of Ref. [47]), the Hawking-Hayward mass has been used only in formal contexts and computed only for analytical solutions of the Einstein equations, but is should be useful also in more practical and astrophysically relevant situations.

We have shown that the spherically symmetric perturbed FLRW metric is equivalent to a McVittie spacetime and we have provided an exact analysis of the turnaround radius in the latter case. We note that the McVittie metric is a good approximation to the FLRW metric near the turnaround radius as defined by our quasi-local mass splitting, which sits well inside the inner apparent horizon at late times when one exists.

Following previous literature, we have considered only spherical symmetry in this work. The limitations of this assumption and the quantitative effects of non-sphericity have been debated in the literature [51]. A background-free discussion without specifying a FLRW background or requiring spherical symmetry should be possible on the lines of Refs. [2]-[5]. The splitting of the Hawking-Hayward quasi-local mass in [47] is general, and will be applied to non-spherical situations in future work.

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